$L_p[0,1] \setminus \bigcup_{q>p} L_q[0,1]$ IS SPACEABLE FOR EVERY $p > 0$

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ABSTRACT. In this short note we prove the result stated in the title; that is, for every $p > 0$ there exists an infinite dimensional closed linear subspace of $L_p[0,1]$ every nonzero element of which does not belong to $\bigcup_{q>p} L_q[0,1]$. This answers in the positive a question raised in 2010 by R. M. Aron on the spaceability of the above sets (for both, the Banach and quasi-Banach cases). We also complete some recent results from [2] for subsets of sequence spaces.

As it has become an standard notion in the last years, given a topological vector space X, we say that a subset $M \subset X$ is *spaceable* (see [1]) if there exists an infinite dimensional closed linear subspace $Y \subset M \cup \{0\}$. Very recently, it was proved in [2] that, for every $p > 0$, the set $\ell_p \setminus \bigcup_{0<q<p} \ell_q$ is spaceable. As a consequence of a lecture delivered by the second author at an international conference held in Valencia (Spain) in the summer of 2010, R. M. Aron asked the question of whether a similar result to [2, Corollary 1.7] would hold for L_p-spaces. The aim of this note is to answer R. M. Aron's question in the positive by means of a constructive procedure and some classical Real Analysis and Linear Algebra techniques.

Theorem. $L_p[0,1] \setminus \bigcup_{q>p} L_q[0,1]$ *is spaceable for every* $p > 0$.

PROOF. Let us first consider the following representation of the semi-open interval $[0,1)$ as a disjoint union of intervals:

$$[0,1) = [0, 1-1/2) \cup [1-1/2, 1-1/4) \cup [1-1/4, 1-1/8) \cup \cdots = \bigcup_{n=1}^{\infty} I_n,$$

where $I_n := [a_n, b_n) = \left[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}\right)$. Notice that, for every $n \in \mathbb{N}$ and every $x \in I_n$, there is a unique $x_n \in [0,1)$ such that

$$x = (1 - x_n)a_n + x_n b_n.$$

Now, given $p > 0$, let us fix a function $f \in L_p[0,1] - \bigcup_{q>p} L_q[0,1]$, and define a sequence of functions $(f_n)_{n=1}^{\infty}$, with $f_n \colon [0,1] \longrightarrow \mathbb{R}$, as follows:

$$f_n(x) = \begin{cases} f(x_n) & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n. \end{cases}$$

The geometric idea is to reproduce the graph of f on the interval I_n. By construction, we have that $\|f_n\|_{L_p} \leq \|f\|_{L_p}$ for every $n \in \mathbb{N}$. Also, the functions f_n are linearly

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independent (they have disjoint supports) and

$$\mathrm{span}\{f_n : n \in \mathbb{N}\} \subset L_p[0,1] \setminus \bigcup_{q>p} L_q[0,1].$$

The latter proves that $L_p[0,1] \setminus \bigcup_{q>p} L_q[0,1]$ is $\aleph_0$-lineable (as it was seen in [3] for the case $p > 1$). In order to go further, the strategy shall be to define a bounded linear and injective operator $T\colon F \longrightarrow L_p[0,1]$, where F is a Banach space, and such that $\overline{T(F)} \cap L_q[0,1] = \{0\}$ for every $q > p$. This shall prove $L_p[0,1] - \bigcup_{q>p} L_q[0,1]$ is spaceable. Indeed, if $(\alpha_j)_{j=1}^{\infty} \in \ell_s$, where $s = 1$ if $p \geq 1$ and $s = p$ if $0 < p < 1$, we have

$$\sum_{n=1}^{\infty} \|\alpha_n f_n\|_{L_p}^s = \sum_{n=1}^{\infty} |\alpha_n|^s \|f_n\|_{L_p}^s \leq \sum_{n=1}^{\infty} |\alpha_n|^s \|f\|_{L_p}^s = \|f\|_{L_p}^s \left\|(\alpha_j)_{j=1}^{\infty}\right\|_s^s < +\infty.$$

Since $L_p[0,1]$ is a Banach space for $p > 1$ and a quasi-Banach space for $0 < p < 1$, it follows that $\sum_{n=1}^{\infty} \alpha_n f_n \in L_p[0,1]$ and, thus,

$$T\colon \ell_s \longrightarrow L_p[0,1] \ , \ T((\alpha_j)_{j=1}^{\infty}) = \sum_{n=1}^{\infty} \alpha_n f_n$$

is a well defined linear operator. Suppose that $\sum_{n=1}^{\infty} \alpha_n f_n = 0$. Then (given any $k \in \mathbb{N}$) we have that, for all $x \in I_k$,

$$\alpha_k f_k(x) = \sum_{n=1}^{\infty} \alpha_n f_n(x) = 0$$

and, since each $f_k \neq 0$, it follows that $\alpha_k = 0$. Hence T is injective and $T(\ell_s)$ is a linear subspace of $L_p[0,1]$. Considering the closure $\overline{T(\ell_s)}$ of $T(\ell_s)$ in $L_p[0,1]$, let us prove that $\overline{T(\ell_s)} \cap L_q[0,1] = \{0\}$ for every $q > p$. Indeed, let $g \in \overline{T(\ell_s)} \setminus \{0\}$. There exist sequences $\left(a_i^{(k)}\right)_{i=1}^{\infty} \in \ell_s$ $(k \in \mathbb{N})$ such that $g = \lim_{k\to\infty} T\left(\left(a_i^{(k)}\right)_{i=1}^{\infty}\right)$ in $L_p[0,1]$. Thus, we have

$$\lim_{k\to\infty} \int_0^1 \left|\sum_{n=1}^{\infty} a_n^{(k)} f_n(x) - g(x)\right|^p dx = 0,$$

which, in particular, implies that

$$\lim_{k\to\infty} \int_0^{\frac{1}{2}} \left|a_1^{(k)} f_1(x) - g(x)\right|^p dx = 0.$$

Next, and by means of a subsequence (if needed), we obtain that

$$\lim_{k\to\infty} a_1^{(k)} f_1(x) = g(x) \text{ a.e. } x \in [0,1/2).$$

Clearly, the set $S = \left\{x \in [0,\frac{1}{2}) : f_1(x) \neq 0\right\}$ is not of measure zero. Thus, given $x' \in S$, we have

$$\lim_{k\to\infty} a_1^{(k)} = \frac{f_1(x')}{g(x')} = \eta \neq 0,$$

concluding that

$$g(x) = \eta f_1(x) \text{ a.e. } x \in [0,1/2).$$

which implies that $g \notin L_q[0, 1/2]$ (regardless of the $q > p$), finishing the proof. ■

Remark. Let us point out that the previous result is the best possible in terms of dimension, since $L_p[0,1]$ is a $\mathfrak{c}$-dimensional linear space (with $\mathfrak{c}$ denoting the continuum). Also, notice that in the above proof, we actually have that $g \notin L_q(I_n)$ regardless of the $q > p$ and $n \in \mathbb{N}$.

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